

INNER PRODUCTS AND MODULE MAPS OF HILBERT C^* -MODULES

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ABSTRACT. Let E and F be two Hilbert C^* -modules over C^* -algebras A and B , respectively. Let T be a surjective linear isometry from E onto F and φ a map from A into B . We will prove in this paper that if the C^* -algebras A and B are commutative, then T preserves the inner products and T is a module map, i.e., there exists a $*$ -isomorphism φ between the C^* -algebras such that

$$\langle Tx, Ty \rangle = \varphi(\langle x, y \rangle),$$

and

$$T(xa) = T(x)\varphi(a).$$

In case A or B is noncommutative C^* -algebra, T may not satisfy the equations above in general. We will also give some condition such that T preserves the inner products and T is a module map.

1. INTRODUCTION

A (right) Hilbert C^* -module over a C^* -algebra A is a right A -module E equipped with A -valued inner product $\langle \cdot, \cdot \rangle$ which is conjugate A -linear in the first variable and A -linear in the second variable such that E is a Banach space with respect to the norm $\|x\| = \|\langle x, x \rangle\|^{1/2}$.

Let X be a locally compact Hausdorff space and H a Hilbert space, the Banach space $C_0(X, H)$ of all continuous H -valued functions vanishing at infinity is a Hilbert C^* -module over the C^* -algebra $C_0(X)$ with inner product $\langle f, g \rangle(x) := \langle f(x), g(x) \rangle$ and module operation $(f\phi)(x) = f(x)\phi(x)$, for all $f \in C_0(X, H)$ and $\phi \in C_0(X)$. Every C^* -algebra A is a Hilbert C^* -module over itself with inner product $\langle a, b \rangle := a^*b$.

Let X and Y be two locally compact Hausdorff spaces, the Banach-Stone theorem states that every surjective linear isometry between $C_0(X)$ and $C_0(Y)$ is a weighted composition operator. More precisely, let T be a surjective linear isometry from $C_0(X)$ onto $C_0(Y)$, then there exists a continuous function $h \in C_0(Y)$ with $|h(y)| = 1$, for all y in Y , and a homeomorphism φ from Y onto X such that T is of the form:

$$(1) \quad Tf(y) = h(y)f(\varphi(y)), \forall f \in C_0(X), \forall y \in Y.$$

Let H_1 and H_2 be two Hilbert spaces. In [7], Jerison characterizes surjective linear isometries between $C_0(X, H_1)$ and $C_0(Y, H_2)$, see also [12, 6]. It is said that every surjective linear isometry T from $C_0(X, H_1)$ onto $C_0(Y, H_2)$ is also of the form (1)

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in which $h(y)$ is a unitary operator from H_1 onto H_2 and h is continuous from Y into $(B(H_1, H_2), SOT)$, the space of all bounded linear operators with the strong operator topology. In this case, we can find a relationship of inner products of $C_0(X, H_1)$ and $C_0(Y, H_2)$ by a simple calculation:

$$\begin{aligned}\langle Tf, Tg \rangle(y) &= \langle Tf(y), Tg(y) \rangle = \langle h(y)(f(\varphi(y))), h(y)(f(\varphi(y))) \rangle \\ &= \langle f(\varphi(y)), f(\varphi(y)) \rangle = \langle f, g \rangle \circ \varphi(y).\end{aligned}$$

i.e.

$$\langle Tf, Tg \rangle = \langle f, g \rangle \circ \varphi.$$

Let $R_\varphi : C_0(X) \rightarrow C_0(Y)$ be the $*$ -isomorphism defined by $R_\varphi(\phi) = \phi \circ \varphi$. Then T preserves the inner products with respect to R_φ , i.e.,

$$\langle Tf, Tg \rangle = R_\varphi(\langle f, g \rangle).$$

By (1), it is easy to see that T is a module map with respect to R_φ in the sense

$$T(f\phi) = T(f)R_\varphi(\phi), \text{ for all } f \in C_0(X, H_1) \text{ and } \phi \in C_0(X).$$

It is natural to ask if these properties are true for surjective linear isometries between Hilbert C^* -modules over C^* -algebras. We will show in this paper that the answer is yes if the C^* -algebras are commutative. Unfortunately, if one of the C^* -algebras is noncommutative, the answer is more complicated. We will give an example (see Example 3) to explain this is not true in general. And we will give a condition on T (see Theorem 9) such that T is a module map and preserves the inner products.

2. PRELIMINARIES

Let E be a Hilbert C^* -module over C^* -algebra A . We set $\langle E, E \rangle$ to be the linear span of elements of the form $\langle x, y \rangle$, $x, y \in E$. E is said to be *full* if the closed two-sided ideal $\overline{\langle E, E \rangle}$ equal A .

A *JB*-triple* is a complex vector space V with a continuous mapping $V^3 \rightarrow V$, $(x, y, z) \rightarrow \{x, y, z\}$, called a *Jordan triple product*, which is symmetric and linear in x, z and conjugate linear in y such that for x, y, z, u, v in V , we have

- (1) $\{x, y, \{z, u, v\}\} = \{\{x, y, z\}, u, v\} - \{z, \{y, x, u\}, v\} + \{z, u, \{x, y, v\}\}$;
- (2) the mapping $z \rightarrow \{x, x, z\}$ is hermitian and has non-negative spectrum;
- (3) $\|\{x, x, x\}\| = \|x\|^3$.

In [5], J. M. Isidro shows that every Hilbert C^* -module is a JB*-triple with the Jordan triple product

$$\{x, y, z\} = \frac{1}{2}(x\langle y, z \rangle + z\langle y, x \rangle).$$

A well-known theorem of Kaup [10] (see also [1]) states that every surjective linear isometry between JB*-triples is a *Jordan triple homomorphism*, i.e., it preserves the Jordan triple product

$$T\{x, y, z\} = \{Tx, Ty, Tz\}, \forall x, y, z \in E.$$

Hence, if T is a surjective linear isometry between Hilbert C^* -modules, then

$$(2) \quad T(x\langle y, z \rangle + z\langle y, x \rangle) = Tx\langle Ty, Tz \rangle + Tz\langle Ty, Tx \rangle, \forall x, y, z \in E.$$

The equation (2) holds if and only if

$$(3) \quad T(x\langle x, x \rangle) = Tx\langle Tx, Tx \rangle, \forall x \in E$$

by triple polarization

$$2\{x, y, z\} = \frac{1}{8} \sum_{\alpha^4=\beta^2=1} \alpha\beta\langle x + \alpha y + \beta z, x + \alpha y + \beta z \rangle(x + \alpha y + \beta z).$$

A ternary ring of operators (TRO) between two Hilbert spaces H and K is a linear subspace \mathfrak{R} of $B(H, K)$, the space of all bounded linear operators from H into K , satisfying $AB^*C \in \mathfrak{R}$. Zettl shows in [17] that every Hilbert C^* -module is isomorphic to a norm closed TRO. In this case, Hilbert C^* -modules have another *triple product*, i.e.,

$$\{x, y, z\} := x\langle y, z \rangle.$$

A map between TROs is said to be a *triple homomorphism* if it preserves the triple products. In the case of Hilbert C^* -modules, a map T is a triple homomorphism if it satisfies

$$(4) \quad T(x\langle y, z \rangle) = Tx\langle Ty, Tz \rangle, \forall x, y, z.$$

We have known every surjective linear isometry is a Jordan triple homomorphism, but it could not be a *triple homomorphism*, see Example 3.

Let \mathcal{R} be a TRO. Then $M_n(\mathcal{R})$, the space of all $n \times n$ matrices whose entries are in \mathcal{R} , has a TRO-structure. Let T be a map between TROs \mathcal{R}_1 and \mathcal{R}_2 . For all positive integer n , define maps $T^{(n)} : M_n(\mathcal{R}_1) \rightarrow M_n(\mathcal{R}_2)$ by $T^{(n)}((x_{ij})_{ij}) = (T(x_{ij}))_{ij}$. We call T n -isometry if $T^{(n)}$ is isometric and *complete isometry* if each $T^{(n)}$ is isometric for all n . It has been shown that a surjective linear isometry between TROs is a triple homomorphism if and only if it is completely isometric. More details about TROs mentioned above, we refer to [17], see also [14, 3]. In fact, Solel shows in [16] that every surjective 2-isometry between two full Hilbert C^* -modules is necessarily completely isometric.

3. RESULTS

Note that in the case of a commutative C^* -algebra $A = C_0(X)$, for some locally compact Hausdorff space X , Hilbert C^* -modules over $C_0(X)$ are the same as Hilbert bundles, or equivalently, continuous fields of Hilbert spaces, over X .

We showed the following theorem in [4].

Theorem 1. *Let E and F be two Hilbert C^* -modules over commutative C^* -algebras $C_0(X)$ and $C_0(Y)$, respectively. Then every surjective linear isometry from E onto F is a weighted composition operator*

$$Tf(y) = h(y)(f(\varphi(y))), \forall f \in E, \forall y \in Y$$

Here, φ is a homeomorphism from Y onto X , $h(y)$ is a unitary operator between the corresponding fibers of E and F , for all y in Y .

By the similar argument discussed in the introduction, we have

Corollary 2. *Every surjective linear isometry between Hilbert C^* -modules over commutative C^* -algebras preserves the inner products and is a module map.*

Now we discuss the case of noncommutative C^* -algebras. From equation (4), it seems that a surjective linear isometry T indicates that T preserves inner products and that T is a module map. We explain this could be not true in general by an example.

Example 3. Given a positive integer n . The Hilbert column space H_c^n is the subspace of $M_n(\mathbb{C})$ consisting of all matrices whose non-zero entries are only in the first column. Similarly, the Hilbert row space is the subspace consisting of matrices whose non-zero entries are only in the first row. Clearly, H_c and H_r are right Hilbert C^* -modules over C^* -algebras \mathbb{C} and $M_n(\mathbb{C})$, respectively, with the inner product $\langle A, B \rangle := A^*B$. Define a surjective linear isometry $T : H_r^n \rightarrow H_c^n$ by $T(A) = A^t$, the transpose of A . Then $\langle T(A), T(B) \rangle = \text{tr} \langle A, B \rangle$, the trace of $\langle A, B \rangle$, but T is not a module map with respect to the trace. For the surjective linear isometry $T : H_c^n \rightarrow H_r^n$, $T(A) = A^t$. Let $\varphi : \mathbb{C} \rightarrow M_n(\mathbb{C})$ be defined by $\varphi(\alpha) = \alpha I$. Then T is a module map with respect to φ , but the equation $\langle TA, TB \rangle = \varphi(\langle A, B \rangle)$ does not hold. It is clear that T does not satisfy the equation (4).

Remark 4. In fact, the corollary above says that there exists a $*$ -isomorphism φ between the C^* -algebras such that

$$\langle Tx, Ty \rangle = \varphi(\langle x, y \rangle)$$

and

$$T(xa) = T(x)\varphi(a).$$

We have seen in the Example 3 that even if T is a module map or preserves the inner products, the map φ might be just a linear map.

In the following, E and F stand for two Hilbert C^* -modules over C^* -algebras A and B , respectively. T is a map from E into F and φ is a map from A into B . The following lemmas explain the relations of T , φ , when T preserves the inner products and when T is a module map, see also [8].

Lemma 5. *If φ is linear, every map T from E into F which preserves the inner products with respect to φ is linear.*

Proof. Since T preserves the inner products with respect to φ . Then for all x, y and z in E , α in \mathbb{C} ,

$$\langle T(\alpha x + y), Tz \rangle = \varphi(\langle \alpha x + y, z \rangle) = \alpha \varphi(\langle x, z \rangle) + \varphi(\langle y, z \rangle) = \langle \alpha Tx + Ty, Tz \rangle.$$

Similarly, we have

$$\langle Tx, T(\alpha y + z) \rangle = \langle Tx, \alpha Ty + Tz \rangle.$$

It is easy to show that

$$\langle T(\alpha x + y) - (\alpha Tx + Ty), T(\alpha x + y) - (\alpha Tx + Ty) \rangle = 0.$$

This proves $T(\alpha x + y) = \alpha Tx + Ty$ and hence T is linear. \square

Lemma 6 ([8]). *Let T be a surjective linear map which preserves the inner products and is a module map w.r.t. φ . If F is full, then φ is a $*$ -homomorphism.*

Proof. Let a_1, a_2 in A and α in \mathbb{C} . It is easy to show that

$$\begin{aligned} & T(x)(\varphi(\alpha a_1 + a_2) - \alpha\varphi(a_1) - \varphi(a_2)) \\ &= T(x)\varphi(\alpha a_1 + a_2) - \alpha T(x)\varphi(a_1) - T(x)\varphi(a_2) \\ &= T(\alpha x a_1 + x a_2) - \alpha T(x a_1) - T(x a_2) = 0. \end{aligned}$$

and

$$\begin{aligned} & T(x)(\varphi(a_1 a_2) - \varphi(a_1)\varphi(a_2)) \\ &= T(x)\varphi(a_1 a_2) - T(x)\varphi(a_1)\varphi(a_2) \\ &= T(x a_1 a_2) - T(x a_1 a_2) = 0. \end{aligned}$$

Since T is surjective and F is full, we have $\varphi(\alpha a_1 + a_2) = \alpha\varphi(a_1) + \varphi(a_2)$ and $\varphi(a_1 a_2) = \varphi(a_1)\varphi(a_2)$.

Let x, y in A , we have

$$\varphi(\langle x, y \rangle^*) = \varphi(\langle y, x \rangle) = \langle Ty, Tx \rangle = \langle Tx, Ty \rangle^* = \varphi(\langle x, y \rangle)^*.$$

For a in A ,

$$\begin{aligned} & \langle T(x)(\varphi(a^*) - \varphi(a)^*), T(x)(\varphi(a^*) - \varphi(a)^*) \rangle \\ &= \varphi(a^*)^* \varphi(\langle x, x \rangle) \varphi(a^*) - \varphi(a^*)^* \varphi(\langle x, x \rangle) \varphi(a)^* - \varphi(a) \varphi(\langle x, x \rangle) \varphi(a^*) + \varphi(a) \varphi(\langle x, x \rangle) \varphi(a)^* \\ &= (\varphi(\langle x a^*, x \rangle) \varphi(a^*))^* - (\varphi(a) \varphi(\langle x, x \rangle) \varphi(a^*))^* - \varphi(\langle x a^*, x a^* \rangle) + (\varphi(a) \varphi(\langle x, x a^* \rangle))^* \\ &= 0. \end{aligned}$$

Hence, $T(x)(\varphi(a^*) - \varphi(a)^*) = 0$ for all x in E . Since T is surjective and F is full, we have $\varphi(a^*) = \varphi(a)^*$. \square

Lemma 7. *If φ is a $*$ -homomorphism, then every map T which preserves the inner products w.r.t. φ is a module map w.r.t. φ .*

Proof. Let x and y in E and a in A . Then

$$\langle T(xa), Ty \rangle = \varphi(\langle xa, y \rangle) = \varphi(a)^* \varphi(\langle x, y \rangle) = \langle T(x)\varphi(a), Ty \rangle.$$

Similarly, we have

$$\langle T(x), T(ya) \rangle = \langle T(x), T(y)\varphi(a) \rangle.$$

It is easy to show that

$$\langle T(xa) - T(x)\varphi(a), T(xa) - T(x)\varphi(a) \rangle = 0.$$

Hence, $T(xa) = T(x)\varphi(a)$. \square

Lemma 8 ([13]). *Let T be a surjective linear isometry and φ a $*$ -isomorphism. If T is a module map w.r.t. φ , then T preserves the inner products with respect to φ .*

Proof. It suffices to prove that $\langle Tx, Tx \rangle = \varphi(\langle x, x \rangle)$ for all x in E . Note that $|a| := (a^*a)^{1/2}$. For all b in B , let $\varphi(a) = b$, then

$$\begin{aligned} & \|Tx|b|\|^2 = \|b^*|Tx|^2b\| = \|\langle T(x)\varphi(a), T(x)\varphi(a) \rangle\| \\ &= \|\langle T(xa), T(xa) \rangle\| = \|\langle xa, xa \rangle\| = \|x|a|\|^2 = \|\varphi(|x|a)\|^2 = \|\varphi(|x|)b\|^2. \end{aligned}$$

By Lemma 3.5 in [11], we get $|Tx| = (\varphi(|x|))$ and hence $\langle Tx, Tx \rangle = \varphi(\langle x, x \rangle)$. \square

Theorem 9. *Let T be a surjective linear 2-isometry from E onto F . Then there exists a $*$ -isomorphism φ from $\overline{\langle E, E \rangle}$ onto $\overline{\langle F, F \rangle}$ such that, for all x, y in E , and a in A ,*

$$\langle Tx, Ty \rangle = \varphi(\langle x, y \rangle)$$

and

$$T(xa) = T(x)\varphi(a).$$

Proof. We can regard E and F as full modules over $\langle E, E \rangle$ and $\langle F, F \rangle$, respectively. In this case, as we mentioned above, T is completely isometric and hence it preserves the triple products

$$T(z\langle x, y \rangle) = Tz\langle Tx, Ty \rangle, \forall x, y, z \in E.$$

Define $\varphi : \langle E, E \rangle \rightarrow \langle F, F \rangle$ by

$$\varphi\left(\sum_{i=1}^n \alpha_i \langle x_i, y_i \rangle\right) := \sum_{i=1}^n \alpha_i \langle Tx_i, Ty_i \rangle, \quad x_i, y_i \in E, \quad \alpha_i \in \mathbb{C}, \quad i = 1, \dots, n.$$

Let x_i, y_i and $z \in E$, $\alpha_i \in \mathbb{C}$, $i = 1, \dots, n$. Then $\sum_{i=1}^n \alpha_i \langle x_i, y_i \rangle = 0$ if and only if $z(\sum_{i=1}^n \alpha_i \langle x_i, y_i \rangle) = 0$ for all z if and only if $T(z)(\sum_{i=1}^n \alpha_i \langle Tx_i, Ty_i \rangle) = \sum_{i=1}^n \alpha_i Tz\langle Tx_i, Ty_i \rangle = \sum_{i=1}^n \alpha_i T(z\langle x_i, y_i \rangle) = T(z(\sum_{i=1}^n \alpha_i \langle x_i, y_i \rangle)) = 0$ for all z if and only if $\sum_{i=1}^n \alpha_i \langle Tx_i, Ty_i \rangle = 0$ since T is injective, $\sum_{i=1}^n \alpha_i \langle x_i, y_i \rangle \in \langle E, E \rangle$ and $\sum_{i=1}^n \alpha_i \langle Tx_i, Ty_i \rangle \in \langle F, F \rangle$. This shows that φ is well-defined and injective. From the definition of φ , since T is surjective, it is clear that φ is a surjective $*$ -homomorphism and T preserves the inner products w.r.t. φ . By lemma 7, T is a module map w.r.t. φ . \square

Corollary 10. *Every surjective linear 2-isometry between two full Hilbert C^* -modules preserves the inner products and is a module map with respect to some $*$ -isomorphism of underlying C^* -algebras.*

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